Solutions to 8th Order Polynomial Minimization Problem *

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Abstract

This paper presents a special canonical dual transformation method for finding global minimizer of a class of 8th order polynomials. The method can be used to solve the 7th order nonlinear algebraic equations that are their derivatives. Applications are illustrated by examples.

1 Primal Problem and its Canonical Dual

This paper presents a method to find global minimizer of the following 8th order polynomial minimization problem

$$P_8(x) = \frac{1}{2}a_2(y_2(x))^2 + b_2 y_2(x) + c_2 - dx, \tag{1}$$

where

$$y_2(x) = \frac{1}{2}a_1(y_1(x))^2 + b_1 y_1(x) + c_1, \ \ y_1(x) = \frac{1}{2}a_0 x^2 + b_0 x + c_0, \tag{2}$$

where $a_i, b_i, c_i (i = 0, 1, 2)$, and d are given constants. In this paper, we assume that $a_i \neq 0$ ($i = 0, 1, 2$). The criticality condition of $P_8(x)$ leads to a 7th order nonlinear algebraic equation

$$(a_2 y_2(x) + b_2)(a_1 y_1(x) + b_1)(a_0 x + b_0) - d = 0. \tag{3}$$

Clearly, direct methods for solving this 7th order nonlinear algebraic equation are very difficult, or even impossible. Also, identifying the global minimizer of $P_8(x)$ is a fundamentally difficult task in global optimization. The canonical duality theory developed in [1] is a potentially useful methodology, and can be used to solve a relatively large class of nonlinear algebraic and differential equations. The associated triality theory can be used to identify both global minimizer and local extrema. In the recent paper [2], a special polynomial $P_n(x)$ with $b_i = 0 (i = 0, 1, 2, \cdots, n)$ has been studied. The purpose of this paper is to generalize this result to solve the 7th order nonlinear equation and to identify global minimizer of $P_8(x)$.

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*The main result of this paper was announced at the 2007 Virginia State Science and Engineering Fair and received an award.

2   Canonical Dual Polynomial

From the pattern of the 8th order polynomial defined in (1), we can let

$$y_2(x) = U_1(y_1(x)) = P_4(x) = \frac{1}{2}a_1(y_1(x))^2 + b_1 y_1(x) + c_1, \tag{4}$$

which is a quadratic function of y_1. The dual variable s_1 of y_1 is defined as

$$s_1 = U_1'(y_1) = a_1 y_1 + b_1. \tag{5}$$

By solving this duality relation, we have

$$y_1(s_1) = \frac{s_1 - b_1}{a_1}. \tag{6}$$

Thus, by using the Legendre transformation

$$U_1^*(s_1) = y_1(s_1)s_1 - U_1(y_1(s_1)), \tag{7}$$

the Legendre conjugate function $U_1^*(s_1)$ is well-defined as

$$
\begin{aligned}
U_1^*(s_1) &= -c_1 - \frac{b_1(s_1 - b_1)}{a_1} + \frac{s_1(s_1 - b_1)}{a_1} - \frac{(s_1 - b_1)^2}{2a_1} \\
&= \frac{(s_1 - b_1)^2}{2a_1} - c_1.
\end{aligned} \tag{8}
$$

In the same way, U_2 is set equal to

$$U_2(y_2(x)) = \frac{1}{2}a_2(y_2(x))^2 + b_2 y_2(x) + c_2. \tag{9}$$

The duality relation between y_2 and s_2 is

$$s_2 = U_2'(y_2) = a_2 y_2 + b_2. \tag{10}$$

Then, the Legendre conjugate of $U_2(y_2)$ is

$$
\begin{aligned}
U_2^*(s_2) &= -c_2 - \frac{b_2(s_2 - b_2)}{a_2} + \frac{s_2(s_2 - b_2)}{a_2} - \frac{(s_2 - b_2)^2}{2a_2} \\
&= \frac{(s_2 - b_2)^2}{2a_2} - c_2.
\end{aligned} \tag{11}
$$

By the fact that

$$U_2(y_2) + U_2^*(s_2) = y_2 s_2, \tag{12}$$

the polynomial (1) can be written as

$$P_8(x) = U_2(x) + c_2 - dx = y_2(x)s_2 - U_2^*(s_2) + c_2 - dx. \tag{13}$$

Manipulation of equations (4) and (7) gives

$$y_2 = U_1 = y_1 s_1 - U_1^*(s_1). \tag{14}$$

Therefore,

$$P_8(x) = s_2(y_1(x)s_1 - U_1^*) - U_2^* + c_2 - dx. \tag{15}$$

Let $\Xi(x, s_1, s_2)$ equal to equation (15). Further simplification yields

$$\Xi(x, s_1, s_2) = s_2[(\frac{1}{2}a_0 x^2 + b_0 x + c_0)s_1 - U_1^*(s_1)] - U_2^*(s_2) + c_2 - dx. \tag{16}$$

By solving $\partial\Xi/\partial x$ we have

$$x(s_1, s_2) = \frac{d}{a_0 s_1 s_2} - \frac{b_0}{a_0}. \tag{17}$$

Substituting this x into (16) and setting the final equation equal to $P^d(s_1, s_2)$, the canonical dual function can be obtained as

$$
\begin{aligned}
P^d(s_1, s_2) &= \Xi(x(s_1, s_2), s_1, s_2) \\
&= s_1 s_2 \left[c_0 - \frac{(\frac{d}{s_1 s_2} - b_0)^2}{2a_0} \right] - s_2 U_1^*(s_1) - U_2^*(s_2) + c_2.
\end{aligned} \tag{18}
$$

By combining (4), (6) and (10) and simplifying, the relation between s_1 and s_2 is given by

$$
\begin{aligned}
s_2 &= a_2 \left(\frac{1}{2}a_1 \left(\frac{s_1 - b_1}{a_1} \right)^2 + b_1 \frac{s_1 - b_1}{a_1} + c_1 \right) + b_2 \\
&= a_2 \left(\frac{s_1^2 - b_1^2}{2a_1} + c_1 \right) + b_2.
\end{aligned} \tag{19}
$$

By substituting (19) into (18) and letting $s = s_1$, the canonical dual function is finally obtained as $P^d(s) = P^d(s, s_2(s))$. By using Mathematica, $P^d(s)$ can be calculated as

$$
\begin{aligned}
P^d(s) &= c_2 - \frac{a_2(s^2 + 2a_1 c_1 - b_1^2)^2}{8a_1^2} + \left(b_2 + a_2 c_1 + \frac{a_2(s^2 - b_1^2)}{2a_1} \right) \\
&\left[c_1 - \frac{(s - b_1)^2}{2a_1} + s \left(c_0 - (2a_0)^{-1} \left(b_0 - \frac{2a_1 d}{s(2a_1 b_2 + a_2(s^2 + 2a_1 c_1 - b_1^2))} \right)^2 \right) \right]
\end{aligned} \tag{20}
$$

From the canonical duality theory [1], we have the following result:

Theorem 1 *If s is a critical point of $P^d(s)$, then*

$$x(s) = \frac{d}{a_0 s_2 s} - \frac{b_0}{a_0}, \tag{21}$$

with

$$s_2 = a_2(\frac{s_1^2 - b_1^2}{2a_1} + c_1) + b_2 \tag{22}$$

is a critical point of $P(x)$ and $P(x) = P^d(s)$. Moreover, if s is the largest critical point, then $x(s)$ is a global minimizer of $P(x)$.

3   Applications

3.1   Example 1

Let $a_0 = .1$; $a_1 = .51$; $a_2 = .5$; $b_0 = 0$; $b_1 = 0$; $b_2 = 0$; $c_0 = -6$; $c_1 = 0$; $c_2 = .5$; $d = 1.5$, we have

$$P_8(x) = 0.5 - 1.5x + 0.0162563(-6 + 0.05x^2)^4. \tag{23}$$

This is a nonconvex polynomial of degree eight with one global minimizer, one local minimizer and one local maximizer as shown in Fig. 1.
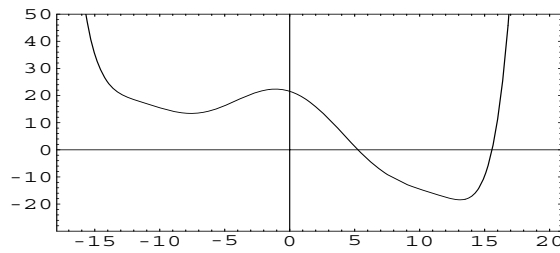

Figure 1. Nonconvex $P_8(x)$ with one global minimizer, one local minimizer and one local maximizer.

The canonical dual of this $P_8(x)$ has a very simple form

$$P^d(s) = 0.5 + 0.490196(-6 - \frac{46.818}{s^6})s^3 - 0.720877s^4. \tag{24}$$

Its graph is shown in Fig. 2.
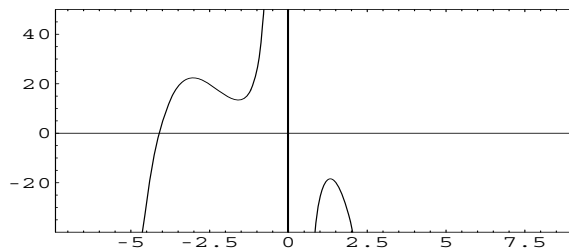
Figure 2. Canonical dual polynomial $P^d(s)$.

The criticality condition $P^d(s) = 0$ has three real roots

$$s_1 = 1.32632 > s_2 = -1.5917 > s_3 = -3.02909. \tag{25}$$

From equation (21), we have $x_1 = 13.1154$, $x_2 = -7.58817$, and $x_3 = -1.10099$. It is easy to check that

$$P(x_1) = -18.4294 = P^d(s_1) < P(x_2) = 13.4246 = P^d(s_2) < P(x_3) = 22.3811 = P^d(s_3). \tag{26}$$

This shows that x_1 is a global minimizer, x_2 is a local minimizer, and x_3 is a local maximizer (see Fig. 3).
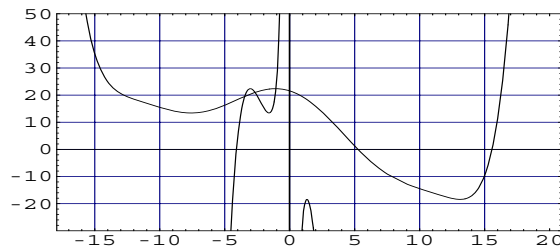

Figure 3. Canonical dual polynomial $P^d(s)$.

3.2 Example 2

In Example 1, if we let $a_0 = .1$; $a_1 = .51$; $a_2 = .5$; $b_0 = -.1$; $b_1 = -.2$; $b_2 = 0$; $c_0 = -7.5$; $c_1 = -5$; $c_2 = .5$; $d = -1.5$, the graph of the polynomial $P_8(x)$ is shown in Fig. 4.

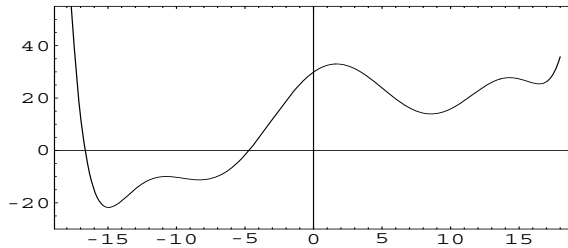Figure 4. Nonconvex $P_8(x)$ with $c_1 = -7.5$.

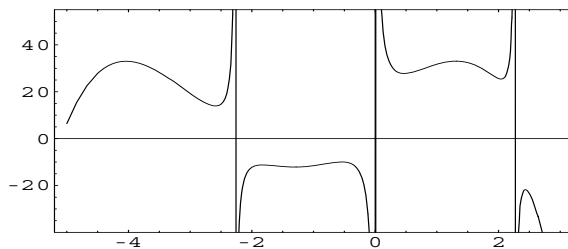In this case, the canonical dual function $P^d(s)$ has a total of seven critical points (see Fig. 5).

Figure 5. Canonical dual polynomial $P^d(s)$.

$$s_7 = -4.03877 < s_6 = -2.58867 < s_5 = -1.82974 < s_4 = -0.537399$$
$$< s_3 = 0.467103 < s_2 = 2.04221 < s_1 = 2.43476.$$

By the Theorem 1 we know that $x_1 = -14.9475$ is a global minimizer of $P(x)$ and $P(x_1) =$
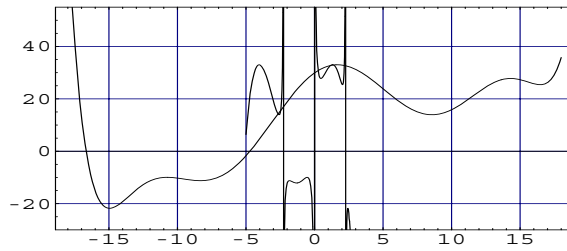
$P^d(s_1) = -21.7721.$(see Fig. 6).

Figure 6. Graphs of $P(x)$ and $P^d(s)$.

4 Conclusions

The results of this paper can be used to find the all critical points of the eighth polynomial derivative of equation (1). One advantage of using this way to find the critical values of P_8 is that, when the derivative of $P^d(s)$ is set to 0 and the critical values of $P^d(s)$ are found, the critical value with the largest value of s is the absolute minimum.

Supplementary File

A Mathematica file is available for those who wish to experiment with these techniques.

Acknowledgements

References

[1] Gao, D.Y.(2000). *Duality Principles in Nonconvex Systems: Theory, Methods and Applications.* Kluwer Academic Publishers, Dordrecht/Boston/London, xviii+454pp.

[2] Gao, D.Y. (2006). Complete solutions to a class of polynomial minimization problems, *J. Global Optimization,* **35,** 131-143.